

6.6 Powers and Roots of Complex Numbers

De Moivre's Theorem

If $z = r(\cos \theta + i \sin \theta)$ is a complex number and n is any positive integer, then

$$z^n = r^n (\cos n\theta + i \sin n\theta)$$

Examples:

a) Simplify $(1+i)^6$.

b) Simplify $(\sqrt{3}-i)^4$.

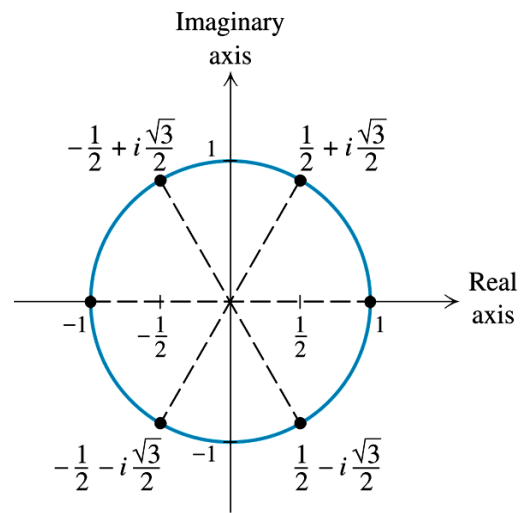
Roots of a Complex Number

How many square roots does 4 have?

How many square roots does -9 have?

How many sixth roots does 1 have?

It turns out that 1 has 6 sixth roots, and they are distributed evenly around the complex plane.



Sixth Roots of Unity

The complex number $a+bi$ is an n th root of the complex number z if $(a+bi)^n = z$.

For any positive integer n , the complex number $z = r(\cos \theta + i \sin \theta)$ has exactly n distinct n th roots given by

the expression $r^{1/n} \left[\cos \left(\frac{\theta + 360^\circ k}{n} \right) + i \sin \left(\frac{\theta + 360^\circ k}{n} \right) \right]$ for $k = 0, 1, 2, \dots, n-1$.

In radians, the roots are given by $r^{1/n} \left[\cos \left(\frac{\theta + 2k\pi}{n} \right) + i \sin \left(\frac{\theta + 2k\pi}{n} \right) \right]$ for $k = 0, 1, 2, \dots, n-1$.

The first of the n roots has an argument of $\frac{\theta}{n}$, and the other roots are spaced $\left(\frac{360}{n} \right)^\circ$ apart.

(The circle is divided evenly into n pieces.)

Examples:

a) Find all of the fourth roots of the complex number $-8-8i\sqrt{3}$.

b) Find all the cube roots of 125.

c) Find all complex solutions to $x^4 - 81 = 0$.